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Polynomials with a Parabolic Majorant and the Duffin–Schaeffer Inequality

Q. I. RAHMAN AND A. O. WATT

Département de Mathématiques et de Statistique, Université de Montréal, Montréal, Québec, Canada H3C3J7

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For $y \in \mathbb{R}$ let $I_y := \{x + iy : -1 \leqslant x \leqslant 1\}$. It was proved by R. J. Duffin and A. C. Schaeffer that if $p(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ is a polynomial of degree at most n with real coefficients such that $|p(\cos(\nu\pi/n))| \leqslant 1$ for $\nu = 0, 1, ..., n$ and T_n is the nth Chebyshev polynomial of the first kind then $\max_{z \in I_y} |p^{(k)}(z)| \leqslant |T_n^{(k)}(1 + iy)|$ for $k = 1, 2, ...$ To this we add that if $\tau_{n+2}(z) := (1 - z^2)\, T_n(z)$ then $\max_{z \in I_y} |(d^k/dz^k)((1 - z^2)\, p(z))| \leqslant |\tau_{n+2}^{(k)}(1 + iy)|$ for $k = 3, 4, ...$ The result can be looked upon as an inequality for polynomials with a parabolic majorant, analogous to that of Duffin and Schaeffer. © 1992 Academic Press, Inc.

1. INTRODUCTION

Let us denote by $\|\cdot\|$ the maximum norm on $[-1, +1]$ and by $\mathscr{P}_n$ the set of all polynomials of degree at most n. For p belonging to $\mathscr{P}_n$ and vanishing at -1, $+1$ let

$$\|p\|_* := \sup_{-1<x<1} \frac{|p(x)|}{\sqrt{1-x^2}}; \qquad \|p\|_{**} := \sup_{-1<x<1} \frac{|p(x)|}{1-x^2}.$$

Further, let $T_n(x) := \cos(n \arccos x)$ be the nth Chebyshev polynomial of the first kind and $U_m(x) := \sin((m+1) \arccos x)/\sin(\arccos x)$ the mth Chebyshev polynomial of the second kind. We also need to introduce the polynomials

$$\upsilon_n(x) := (1 - x^2)\, U_{n-2}(x), \qquad \tau_n(x) := (1 - x^2)\, T_{n-2}(x).$$

Let $p \in \mathscr{P}_n$. According to a classical result of W. A. Markoff [2]

$$\|p^{(k)}\| \leqslant T_n^{(k)}(1) \qquad \text{for all} \quad k \in \mathbb{N} \qquad \text{if} \quad \|p\| \leqslant 1. \tag{1}$$

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It is also known [6, 3] that

$$\|p^{(k)}\| \leqslant |v_n^{(k)}(1)| \qquad \text{for all} \quad k \in \mathbb{N} \qquad \text{if} \quad \|p\|_* \leqslant 1; \tag{2}$$

$$\|p^{(k)}\| \leqslant |\tau_n^{(k)}(1)| \qquad \text{for} \quad k=2, 3, ... \qquad \text{if} \quad \|p\|_{**} \leqslant 1. \tag{3.1}$$

As regards the missing case $k=1$, when $\|p\|_{**} \leqslant 1$ we have [5]

$$\|p'\| \leqslant |\tau_n'(1)| \qquad \text{if} \quad n=4, \tag{3.2}$$

$$\|p'\| \leqslant |\tau_n'(0)| \qquad \text{for } odd \quad n \geqslant 5, \tag{3.3}$$

whereas for *even* n

$$\|p'\| \leqslant n-2-\frac{\pi^2}{8n}+O(n^{-2}) \qquad \text{as} \quad n \to \infty. \tag{3.4}$$

Here it may be added that $|\tau_n'(\pi/2(n-2))| = n-2-\pi^2/8n+O(n^{-2})$ as $n \to \infty$.

A remarkable generalization of (1) was found by Duffin and Schaeffer who proved (see [1, Theorem II] or [8, pp. 130–138]):

THEOREM A. *Let $p \in \mathscr{P}_n$. If $p(x)$ is real for real x and if*

$$\left| p\left(\cos\frac{\nu\pi}{n}\right)\right| \leqslant 1 \qquad \textit{for} \quad \nu=0, 1, ..., n, \tag{4}$$

then for $k \in \mathbb{N}$

$$|p^{(k)}(x+iy)| \leqslant |T_n^{(k)}(1+iy)|, \qquad -1 \leqslant x \leqslant 1, \quad -\infty < y < \infty. \tag{5}$$

The corresponding extension of (2) which was obtained in [7] reads as follows:

THEOREM B. *Let*

$$\xi_0 := 1, \quad \xi_n := -1, \quad \textit{and} \quad \xi_\nu := \cos\left(\frac{2\nu-1}{n-1}\frac{\pi}{2}\right), \qquad \nu=1, ..., n-1. \tag{6}$$

If $p \in \mathscr{P}_n$ such that

$$|p(\xi_\nu)| \leqslant (1-\xi_\nu^2)^{1/2} \qquad \textit{for} \quad \nu=0, 1, ..., n, \tag{7}$$

then

$$\|p^{(k)}\| \leqslant ||v_n^{(k)}(1)| \qquad \textit{for} \quad k=2, 3, ... \tag{8}$$

whereas

$$\|p'\| \leqslant (n-1)\left(\frac{2}{\pi}\log(n-1)+3\right). \tag{9}$$

Further, if $p(x)$ *is real for real* x *then*

$$|p^{(k)}(x+iy)| \leqslant |v_n^{(k)}(1+iy)| \quad \textit{for} \quad (x,y)\in[-1,1]\times\mathbb{R} \quad \textit{and} \quad k=2,3.... \tag{8'}$$

In (8), (8′) equality holds if and only if $p(x)\equiv \gamma v_n(x)$ where $|\gamma|=1$. Besides, the number $2/\pi$ appearing on the right hand side of (9) cannot be replaced by any smaller number not depending on n.

Here we prove

THEOREM 1. *For given* $n\geqslant 3$, *let*

$$\lambda_v=\lambda_{n,v}:=\cos\left(\frac{v\pi}{n-2}\right), \qquad v=0,1,...,n-2. \tag{10}$$

If $p(x):=(1-x^2)\,q(x)$ *is a polynomial of degree at most n such that*

$$|q(\lambda_v)|\leqslant 1 \qquad \textit{for} \quad v=0,1,...,n-2 \tag{11}$$

then

$$\|p^{(k)}\| \leqslant |\tau_n^{(k)}(1)| \qquad \textit{for} \quad k=3,4,.... \tag{12}$$

Further, if $p(x)$ *is real for real* x, *then*

$$|p^{(k)}(x+iy)| \leqslant |\tau_n^{(k)}(1+iy)| \quad \textit{for} \quad (x,y)\in[-1,1]\times\mathbb{R} \textit{ and } k=3,4.... \tag{12'}$$

2. AUXILIARY RESULTS

We prove Theorem 1 by an argument analogous to that of Duffin and Schaeffer [1]. However, certain details become considerably harder and some new properties of T_n need to be proved. The first two lemmas are taken from [1].

LEMMA 1 [1, Lemma 1]. *If*

$$P(z)=c\prod_{v=1}^{n}(z-x_v)$$

is a polynomial of degree n with n distinct real zeros and if p is a polynomial of degree at most n such that

$$|p'(x_\nu)| \leqslant |P'(x_\nu)| \qquad (\nu = 1, ..., n),$$

then for $k = 1, ..., n$

$$|p^{(k)}(x)| \leqslant |P^{(k)}(x)|$$

at the roots of $P^{(k-1)}(x) = 0$.

LEMMA 2 [1, Theorem I]. *Let P be a polynomial of degree n with n distinct real zeros to the left of the point* 1 *and suppose that*

$$|P(x+iy)| \leqslant |P(1+iy)| \qquad \textit{for} \quad (x, y) \in [-1, 1] \times \mathbb{R}.$$

If p is a polynomial of degree at most n with real coefficients such that

$$|p'(x)| \leqslant |P'(x)| \qquad \textit{whenever} \quad P(x) = 0,$$

then for $k = 1, 2, ..., n$

$$|p^{(k)}(x+iy)| \leqslant |P^{(k)}(1+iy)| \qquad \textit{for} \quad (x, y) \in [-1, 1] \times \mathbb{R}.$$

The next result is needed to prove a new property of T_m contained in Lemma 4.

LEMMA 3. *If p is a polynomial of degree m having all its zeros in* $\operatorname{Im} z > 0$, *then for* $\xi \geqslant 0$

$$(m^2+2)\, p(z) + 3(z - i\xi)\, p'(z)$$

has all its zeros in $\operatorname{Im} z > 0$.

Proof. Let $z_\mu := x_\mu + iy_\mu$ $(\mu = 1, ..., m)$ be the zeros of p. Further, let $z = x + iy$, $x \in \mathbb{R}$, $y \in \mathbb{R}$. Then for $y \leqslant 0$

$$\operatorname{Im}\left\{\frac{p'(z)}{p(z)}\right\} = \sum_{\mu=1}^{m} \operatorname{Im} \frac{1}{x - x_\mu + i(y - y_\mu)} = \sum_{\mu=1}^{m} \frac{-(y - y_\mu)}{|z - z_\mu|^2} > 0;$$

if $\xi \geqslant 0$ and $z - i\xi \neq 0$ then for $y \leqslant 0$

$$\operatorname{Im}\left\{-\frac{m^2+2}{3(z-i\xi)}\right\} = \frac{m^2+2}{3}\,\frac{y-\xi}{|z-i\xi|^2} \leqslant 0.$$

Hence if $\xi \geqslant 0$, then $-(m^2+2)/3(z-i\xi) \neq p'(z)/p(z)$ for $\operatorname{Im} z \leqslant 0$ provided $z - i\xi \neq 0$, i.e., $(m^2+2)\, p(z) + 3(z-i\xi)\, p'(z) \neq 0$ for $\operatorname{Im} z \leqslant 0$ and all $\xi \geqslant 0$

except possibly when $z-i\xi=0$. But if $z-i\xi=0$ then $(m^2+2)\,p(z)+3(z-i\xi)\,p'(z)$ reduces to $(m^2+2)\,p(z)$, which is $\neq 0$ for $\operatorname{Im} z\leqslant 0$, by hypothesis.

LEMMA 4. *The polynomial* $\tau_{m+2}(z):=(1-z^2)\,T_m(z)$ *satisfies*

$$|\tau''_{m+2}(x+iy)|\leqslant|\tau''_{m+2}(1+iy)| \qquad \text{for} \quad (x,y)\in[-1,1]\times\mathbb{R}.$$

Proof. First we note that

$$\begin{aligned}\tau''_{m+2}(z)&=(1-z^2)\,T''_m(z)-4zT'_m(z)-2T_m(z)\\&=zT'_m(z)-m^2T_m(z)-4zT'_m(z)-2T_m(z)\\&=-3zT'_m(z)-(m^2+2)\,T_m(z).\end{aligned}$$

Now let $\xi\in[0,1]$. Then for $x\in[\xi,\infty)$,

$$|T_m(x+iy)|\leqslant|T_m(1+x-\xi+iy)|.$$

Hence

$$R_\alpha(z):=\alpha T_m(z)+T_m(1-\xi+z)$$

does not vanish in the half-plane $\{z\in\mathbb{C}:\operatorname{Re} z\geqslant\xi\}$ whenever $|\alpha|<1$. Applying Lemma 3 to the polynomial $R_\alpha(iz+\xi)$ we conclude that $(m^2+2)\,R_\alpha(iz+\xi)+3(iz+\xi)\,R'_\alpha(iz+\xi)$ does not vanish for $\operatorname{Im} z\leqslant 0$, i.e.,

$$\begin{aligned}&\alpha\{(m^2+2)\,T_m(z)+3zT'_m(z)\}\\&\qquad+(m^2+2)\,T_m(1-\xi+z)+3zT'_m(1-\xi+z)\neq 0\end{aligned}$$

for $\operatorname{Re} z\geqslant\xi$ and $|\alpha|<1$. Setting $z=\xi+iy$ this implies

$$\begin{aligned}|\tau''_{m+2}(\xi+iy)|&\equiv|(m^2+2)\,T_m(\xi+iy)+3(\xi+iy)\,T'_m(\xi+iy)|\\&\leqslant|(m^2+2)\,T_m(1+iy)+3(\xi+iy)\,T'_m(1+iy)|.\end{aligned}\tag{13}$$

Obviously

$$w:=\frac{3T'_m(1+iy)}{(m^2+2)\,T_m(1+iy)}$$

is a point in the right half-plane. Therefore

$$|1+(\xi+iy)\,w|\leqslant|1+(1+iy)\,w|$$

and hence the right-hand side of (13) is majorized by

$$|(m^2+2)\,T_m(1+iy)+3(1+iy)\,T'_m(1+iy)| \equiv |\tau''_{m+2}(1+iy)|.$$

Since $|\tau''_{m+2}(-z)| \equiv |\tau''_{m+2}(z)| \equiv |\tau''_{m+2}(\bar{z})|$ the inequality

$$|\tau''_{m+2}(\xi+iy)| \leqslant |\tau''_{m+2}(1+iy)|$$

also holds for $\xi \in [-1, 0)$.

2.1. *Lower Bounds for $|T_m(x)|$ at the Zeros of τ'_{m+2}*

Given $m \in \mathbb{N}$, let $\lambda_\mu = \lambda_{m,\mu} := \cos \mu\pi/m$ $(\mu = 0, 1, ..., m)$. The zeros of τ'_{m+2} all lie in $(-1, 1)$ and are symmetrically situated with respect to the origin. Denoting them in decreasing order by ξ_μ $(\mu = 0, 1, ..., m)$ we easily see that $\xi_\mu \in (\cos(2\mu+1)\pi/2m, \lambda_\mu)$ for $\mu = 0, ..., [(m-1)/2]$ and that $\xi_{m/2} = 0$ in case m is even. With each ξ_μ we associate the quantity

$$\theta_\mu = \theta_{m,\mu} := \sqrt{\frac{m^2(1-\xi_\mu^2)}{m^2(1-\xi_\mu^2)+4\xi_\mu^2}}.$$

Using

$$(1-\xi_\mu^2)\,T'_m(\xi_\mu) = 2\xi_\mu\,T_m(\xi_\mu)$$

in conjunction with the identity

$$(1-x^2)\,\{T'_m(x)\}^2 + m^2\{T_m(x)\}^2 \equiv m^2$$

we obtain that

$$|T_m(\xi_\mu)| = \theta_\mu \qquad (\mu = 0, 1, ..., m).$$

In the next lemma we obtain a lower bound for θ_μ which is not sharp but is adequate for our purpose.

LEMMA 5. *Let $m \geqslant 3$. For $\mu = 1, ..., m-1$*

$$\theta_\mu > .826674148. \tag{14}$$

Proof. For each m, θ_μ is a decreasing function of $|\xi_\mu|$ and so it is enough to prove (14) for $\mu = 1$. Simple calculation shows that $\theta_1 = .957214044...$ if $m = 3$ whereas $\theta_1 = .924950591...$ if $m = 4$. So let $m \geqslant 5$. Clearly

$$\xi_1 < \lambda_1 = \cos\frac{\pi}{m} < 1 - \frac{\pi^2}{2m^2} + \frac{\pi^4}{24m^4} \leqslant 1 - \frac{4.772448}{m^2}. \tag{15}$$

Hence for $m \geqslant 5$ we have $m^2(1-\xi_1^2) > 8.633845604$ which in turn implies that

$$\theta_1 > \sqrt{\frac{8.633845604}{12.633845604}} = .826674148.$$

Now we need to estimate $\lambda_\mu - \xi_\mu$ from below. This is done in

LEMMA 6. *For* $\mu = 1, ..., [(m-1)/2]$ *we have*

$$\lambda_\mu - \xi_\mu > (3\theta_\mu - 1)\frac{\xi_\mu}{m^2} = \frac{2\xi_\mu}{m^2} - 3(1-\theta_\mu)\frac{\xi_\mu}{m^2}. \tag{16}$$

Proof. We have

$$\begin{aligned}
-2\lambda_\mu T_m(\lambda_\mu) &= [(1-x^2)\,T_m'(x) - 2xT_m(x)]_{\xi_\mu}^{\lambda_\mu} \\
&= \int_{\xi_\mu}^{\lambda_\mu} \{(1-x^2)\,T_m''(x) - 4xT_m'(x) - 2T_m(x)\}\,dx \\
&= \int_{\xi_\mu}^{\lambda_\mu} \{-3xT_m'(x) - (m^2+2)\,T_m(x)\}\,dx \\
&= [-3xT_m(x)]_{\xi_\mu}^{\lambda_\mu} - (m^2-1)\int_{\xi_\mu}^{\lambda_\mu} T_m(x)\,dx \\
&= -3\lambda_\mu T_m(\lambda_\mu) + 3\xi_\mu T_m(\xi_\mu) - (m^2-1)\int_{\xi_\mu}^{\lambda_\mu} T_m(x)\,dx
\end{aligned}$$

and so

$$\begin{aligned}
&3\xi_\mu T_m(\xi_\mu) - \xi_\mu \operatorname{sgn}(T_m(\lambda_\mu)) \\
&\quad = \lambda_\mu T_m(\lambda_\mu) - \xi_\mu \operatorname{sgn}(T_m(\lambda_\mu)) + (m^2-1)\int_{\xi_\mu}^{\lambda_\mu} T_m(x)\,dx.
\end{aligned}$$

Since $|\int_{\xi_\mu}^{\lambda_\mu} T_m(x)\,dx| \leqslant \lambda_\mu - \xi_\mu$ and $\theta_\mu > \frac{1}{3}$ it follows that

$$(3\theta_\mu - 1)\,\xi_\mu < \lambda_\mu - \xi_\mu + (m^2-1)(\lambda_\mu - \xi_\mu),$$

which is what we wanted to prove.

At this stage it is important to obtain a good upper bound for ξ_μ^2.

LEMMA 7. *Let* $m \geqslant 2$. *For* $\mu = 1, ..., m-1$

$$\xi_\mu^2 < 1 - \frac{10}{m^2+10} \tag{17}$$

and so

$$\delta_\mu := \frac{4\xi_\mu^2}{m^2(1-\xi_\mu^2)} \leqslant \frac{2}{5}. \tag{18}$$

Proof. We need to prove (17) only for $\mu = 1$. If $m = 2$, then $\xi_1 = 0$ and so (17) holds. It is a matter of simple calculation that $\xi_1^2 = .170563828 < 9/19 = 1 - 10/(m^2+10)$ if $m = 3$ whereas $\xi_1^2 = .403143528 < 8/13 = 1 - 10/(m^2+10)$ if $m = 4$. Now let $m \geqslant 5$. From (15) and (16) it follows that

$$\xi_1 < 1 - \frac{4.772448}{m^2} - \frac{1}{m^2}(3\theta_1 - 1)\,\xi_1.$$

Since $\theta_1 \geqslant .826674148$ we get

$$\begin{aligned}\xi_1 &< \frac{1 - 4.772448/m^2}{1 + 1.480022444/m^2} < 1 - \frac{6.252470444}{m^2} + \frac{9.253796588}{m^4}\\ &\leqslant 1 - \frac{5.882318581}{m^2}.\end{aligned}$$

Hence

$$\xi_1^2 \leqslant 1 - \frac{10.38057029}{m^2} < 1 - \frac{10}{m^2+10}.$$

This proves (17). As regards (18), it is a direct consequence of (17).

We use (18) to obtain a crucial lower bound for θ_μ depending on δ_μ.

LEMMA 8. *For* $\mu = 1, ..., [(m-1)/2]$

$$\theta_\mu \geqslant 1 - \tfrac{1}{2}\delta_\mu + \tfrac{1}{4}\delta_\mu^2.$$

Proof. According to Taylor's theorem

$$\begin{aligned}\theta_\mu &= \frac{1}{\sqrt{1+\delta_\mu}} =: \theta(\delta_\mu) = \theta(0) + \delta_\mu\theta'(0) + \frac{1}{2!}\delta_\mu^2\theta''(0) + \frac{1}{3!}\delta_\mu^3\theta'''(0)\\ &\quad + \frac{1}{4!}\delta_\mu^4\theta^{(iv)}(\delta') \qquad \text{where} \quad 0 \leqslant \delta' \leqslant \delta_\mu\\ &= 1 - \tfrac{1}{2}\delta_\mu + \tfrac{3}{8}\delta_\mu^2 - \tfrac{5}{16}\delta_\mu^3 + \tfrac{35}{128}\delta_\mu^4(1+\delta')^{-9/2}\\ &> 1 - \tfrac{1}{2}\delta_\mu + \tfrac{3}{8}\delta_\mu^2 - \tfrac{5}{16}\delta_\mu^3\\ &\geqslant 1 - \tfrac{1}{2}\delta_\mu + \tfrac{3}{8}\delta_\mu^2 - \tfrac{1}{8}\delta_\mu^2 \qquad \text{by (18)}\\ &= 1 - \tfrac{1}{2}\delta_\mu + \tfrac{1}{4}\delta_\mu^2.\end{aligned}$$

2.2. *The Sign of* $((1-x^2)^2 T'_m(x)/(x-\lambda_\mu))''$ *at a zero of* τ'_{m+2}

LEMMA 9. *Let* ξ *be a zero of* τ'_{m+2}. *Then for* $\mu = 0, 1, ..., m$

$$\frac{1-\xi^2}{T_m(\xi)} \frac{d^2}{dx^2} \left\{ \frac{(1-x^2)^2 T'_m(x)}{x-\lambda_\mu} \right\} \bigg|_{x=\xi} = \frac{\phi(\xi, \lambda_\mu)}{(\xi-\lambda_\mu)^4},$$

where

$$\phi(\xi, t) := (\xi - t)\{3\xi((m^2-4)(1-\xi^2)+2)(t-\xi)^2 \\ -2(1-\xi^2)(m^2(1-\xi^2)+6\xi^2)(t-\xi)+4\xi(1-\xi^2)^2\}.$$

Proof. It is a matter of simple calculation that

$$\frac{d}{dx}\left\{\frac{(1-x^2)^2 T'_m(x)}{x-\lambda_\mu}\right\}$$

$$= \frac{\{(1-x^2)^2 T''_m(x) - 4x(1-x^2) T'_m(x)\}(x-\lambda_\mu) - (1-x^2)^2 T'_m(x)}{(x-\lambda_\mu)^2}$$

$$= -\frac{(1-3\lambda_\mu x + x^2 + 3\lambda_\mu x^3 - 2x^4) T'_m(x) + m^2(-\lambda_\mu + x + \lambda_\mu x^2 - x^3) T_m(x)}{(x-\lambda_\mu)^2}$$

and

$$(1-x^2)\frac{d^2}{dx^2}\left\{\frac{(1-x^2)^2 T'_m(x)}{x-\lambda_\mu}\right\}$$

$$= -\frac{A(x)(1-x^2) T'_m(x) - B(x)(1-x^2) T_m(x)}{(x-\lambda_\mu)^4},$$

where

$$A(x) := (x-\lambda_\mu)^3 (m^2+3-(m^2+6)x^2) \\ -2(x-\lambda_\mu)(1-3\lambda_\mu x + 2x^2)(1-x^2),$$
$$B(x) := (x-\lambda_\mu)^3 5m^2 x + (x-\lambda_\mu)^2 2m^2(1-x^2).$$

At a zero ξ of τ'_{m+2} we have $(1-\xi^2) T'_m(\xi) = 2\xi T_m(\xi)$ and so setting

$$A_1(\xi) := 3\xi\{(m^2-4)(1-\xi^2)+2\},$$
$$A_2(\xi) := 2(1-\xi^2)\{m^2(1-\xi^2)\{m^2(1-\xi^2)+6\xi^2\},$$
$$A_3(\xi) := 4\xi(1-\xi^2)^2$$

we get

$$(1-\xi^2)\frac{d^2}{dx^2}\left\{\frac{(1-x^2)^2\,T_m'(x)}{x-\lambda_\mu}\right\}\Bigg|_{x=\xi}$$
$$=\frac{A_1(\xi)(\xi-\lambda_\mu)^3+A_2(\xi)(\xi-\lambda_\mu)^2+A_3(\xi)(\xi-\lambda_\mu)}{(\xi-\lambda_\mu)^4}\,T_m(\xi)$$
$$=\frac{\phi(\xi,\lambda_\mu)}{(\xi-\lambda_\mu)^4}\,T_m(\xi).$$

Remark 1. It is important to note that for $\xi=0$, which is one of the zeros of τ_{m+2}' when m is even,

$$\phi(\xi,\lambda_\mu)=\phi(0,\lambda_\mu)=2m^2\lambda_\mu^2\geqslant 0 \qquad \text{for} \quad \mu=0,1,...,m.$$

We claim that $\phi(\xi_\mu,\lambda_\nu)\geqslant 0$ for $\mu=0,1,...,m$ and $\nu=0,1,...,m$. This crucial fact is established in the next four lemmas. The proof which makes use of Lemmas 5–8 is long and tedious. The difficulty lies in the fact that $\phi(\xi_\mu,t)$ changes sign in $(-1,1)$ except when m is even and $\mu=m/2$.

LEMMA 10. *The function $\phi(\xi,t)$ has a zero in $(1,\infty)$ if $0<\xi<1$.*

Proof. Since $\phi(\xi,t)\to-\infty$ as $t\to+\infty$ it suffices to verify that

$$\phi(\xi,1)>0. \tag{19}$$

As is easily seen,

$$\phi(\xi,1)=(1-\xi)^3\,g(\xi),$$

where

$$g(\xi):=(m^2-4)\xi^3-2(m^2-2)\xi^2-(m^2-2)\xi+2m^2,$$

and so it is enough to check that $g(\xi)>0$ for $0<\xi<1$. Indeed, if $m=1$ then $g(\xi)=-3\xi^3+2\xi^2+\xi+2>2$, whereas if $m=2$, then $g(\xi)=-4\xi^2-2\xi+8>2$. In case $m\geqslant 3$ we get the desired conclusion by noting that $g(-2)=-12m^2+44<0$, $g(-1)=6>0$, $g(1)=2>0$, $g(2)=-12<0$, $g(t)\to+\infty$ as $t\to+\infty$.

LEMMA 11. *For $\mu=1,...,[(m-1)/2]$*

$$\phi\left(\xi_\mu,\xi_\mu+(3\theta_\mu-1)\frac{\xi_\mu}{m^2}\right)\geqslant 0.$$

Proof. We have to verify that if

$$L(\xi,t):=\frac{\phi(\xi,t)}{t-\xi}$$

then

$$L\left(\xi_\mu, \xi_\mu + (3\theta_\mu - 1)\frac{\xi_\mu}{m^2}\right) \geqslant 0.$$

We have

$$\begin{aligned} L &= L\left(\xi_\mu, \xi_\mu + (3\theta_\mu - 1)\frac{\xi_\mu}{m^2}\right) \\ &= -\frac{3\xi_\mu^3}{m^4}\{m^2(1-\xi_\mu^2) - 4(1-\xi_\mu^2) + 2\}\{4 - 12(1-\theta_\mu) + 9(1-\theta_\mu)^2\} \\ &\quad -\frac{6}{m^2}(1-\theta_\mu)\,\xi_\mu(1-\xi_\mu^2)\{m^2(1-\xi_\mu^2) + 6\xi_\mu^2\} + \frac{24}{m^2}\xi_\mu^3(1-\xi_\mu^2) \\ &= \frac{12}{m^2}\xi_\mu^3(1-\xi_\mu^2) - \frac{12}{m^4}\xi_\mu^3\{-4(1-\xi_\mu^2) + 2\} \\ &\quad + \frac{36}{m^4}(1-\theta_\mu)\,\xi_\mu^3\{m^2(1-\xi_\mu^2) - 4(1-\xi_\mu^2) + 2\} \\ &\quad - \frac{27}{m^4}(1-\theta_\mu)^2\,\xi_\mu^3\{m^2(1-\xi_\mu^2) - 4(1-\xi_\mu^2) + 2\} \\ &\quad - \frac{6}{m^2}(1-\theta_\mu)\,\xi_\mu(1-\xi_\mu^2)\{m^2(1-\xi_\mu^2) + 6\xi_\mu^2\}. \end{aligned}$$

By Lemma 8

$$1 - \theta_\mu \leqslant \frac{2\xi_\mu^2}{m^2(1-\xi_\mu^2)} - \frac{1}{4}\frac{16\xi_\mu^4}{m^4(1-\xi_\mu^2)^2} = \frac{2\xi_\mu^2}{m^2(1-\xi_\mu^2)} - \frac{4\xi_\mu^4}{m^4(1-\xi_\mu^2)^2}.$$

Hence

$$\begin{aligned} L &\geqslant \frac{12}{m^2}\xi_\mu^3(1-\xi_\mu^2) + \frac{48}{m^4}\xi_\mu^3(1-\xi_\mu^2) - \frac{24}{m^4}\xi_\mu^3 \\ &\quad - \frac{144}{m^4}(1-\theta_\mu)\,\xi_\mu^3(1-\xi_\mu^2) + \frac{72}{m^4}(1-\theta_\mu)\,\xi_\mu^3 \\ &\quad - \frac{27}{m^4}(1-\theta_\mu)^2\xi_\mu^3\{m^2(1-\xi_\mu^2) - 4(1-\xi_\mu^2) + 2\} \\ &\quad - \left\{\frac{2\xi_\mu^2}{m^2(1-\xi_\mu^2)} - \frac{4\xi_\mu^4}{m^4(1-\xi_\mu^2)^2}\right\} 6\xi_\mu(1-\xi_\mu^2)^2 \end{aligned}$$

$$=\frac{24}{m^4}\xi_\mu^3(1-\xi_\mu^2)-\frac{9}{m^4}(1-\theta_\mu)$$

$$\times\xi_\mu^3\{16(1-\xi_\mu^2)-8+3(1-\theta_\mu)(m^2(1-\xi_\mu^2)-4(1-\xi_\mu^2)+2)\}$$

$$\geqslant\frac{24}{m^4}\xi_\mu^3(1-\xi_\mu^2)-\frac{9}{m^4}(1-\theta_\mu)$$

$$\times\xi_\mu^3\{16(1-\xi_\mu^2)-8+\frac{6\xi_\mu^2}{m^2(1-\xi_\mu^2)}(m^2(1-\xi_\mu^2)-4(1-\xi_\mu^2)+2)\}$$

by Lemma 8

$$=\frac{24}{m^4}\xi_\mu^3(1-\xi_\mu^2)-\frac{9}{m^4}(1-\theta_\mu)\,\xi_\mu^3(8-10\xi_\mu^2-\frac{24}{m^2}\xi_\mu^2+3\delta_\mu)$$

$$\geqslant\frac{24}{m^4}\xi_\mu^3(1-\xi_\mu^2)-\frac{9}{m^4}(1-\theta_\mu)\,\xi_\mu^3\left(8-10\xi_\mu^2+\frac{6}{5}\right)\quad\text{by (18)}$$

$$\geqslant\frac{24}{m^4}\xi_\mu^3(1-\xi_\mu^2)-\frac{90}{m^4}(1-\theta_\mu)\,\xi_\mu^3(1-\xi_\mu^2)$$

$$=\frac{90}{m^4}\left(\theta_\mu-\frac{11}{15}\right)\xi_\mu^3(1-\xi_\mu^2)$$

$$\geqslant 0$$

by Lemma 5.

LEMMA 12. *For* $\mu=0, 1, ..., [(m-1)/2]$ *and* $\nu=0, 1, ..., m$

$$\phi(\xi_\mu, \lambda_\nu)\geqslant 0. \tag{20}$$

Proof. From Lemma 10 we know that $\phi(\xi_\mu, t)$ has a zero in $(1, \infty)$. Besides, $\phi(\xi_\mu, t)$ has a zero at ξ_μ with

$$\frac{d}{dt}\phi(\xi_\mu, t)|_{t=\xi_\mu}=-4\xi_\mu(1-\xi_\mu^2)<0.$$

Hence if $\mu=1, ..., [(m-1)/2]$ then, in view of Lemma 11, $\phi(\xi_\mu, t)$ must have a zero in $(\xi_\mu, \xi_\mu+(3\theta_\mu-1)\,\xi_\mu/m^2)$ as well. Being a polynomial of degree 3 in t the function $\phi(\xi_\mu, t)$ has no other zeros and indeed should be positive on $[-1, \xi_\mu)\cup(\xi_\mu+(3\theta_\mu-1)\,\xi_\mu/m^2, 1]$. It follows from Lemma 6 that the interval $[\lambda_\mu, 1]$ is contained in $[\xi_\mu+(3\theta_\mu-1)\,\xi_\mu/m^2, 1]$ and so $\phi(\xi_\mu, t)\geqslant 0$ for $t\in[-1, \xi_\mu]\cup[\lambda_\mu, 1]$. This proves (20) for

$\mu=1, ..., [(m-1)/2]$. We can argue the same way in the case $\mu=0$; although Lemma 11 is not available, (19) serves the purpose.

More generally, we have

LEMMA 12′. (20) *holds for* $\mu=0, 1, ..., m$ *and* $\nu=0, 1, ..., m$.

Proof. That (20) holds for $\mu=m/2$ when m is even was pointed out in Remark 1. It also holds for $\mu=[(m+1)/2], ..., m$ since

$$\phi(\xi, t) \equiv \phi(-\xi, -t)$$

and

$$\xi_\mu = -\xi_{m-\mu} \quad (\mu=0, 1, ..., m), \qquad \lambda_\nu = -\lambda_{m-\nu} \quad (\nu=0, 1, ..., m).$$

Now we are ready to prove

LEMMA 13. *Let* $p(x) := (1-x^2)\, q(x)$ *be a polynomial of degree at most* n *such that* $|q(x)| \leqslant 1$ *at* $\lambda_\nu = \cos(\nu\pi/(n-2))$ $(\nu=0, 1, ..., n-2)$. *Then at the roots of* $\tau_n'(x)=0$

$$|p''(x)| \leqslant |\tau_n''(x)|.$$

The equality can occur only if $p(x) \equiv \gamma\tau_n(x)$ *for some constant* γ, $|\gamma|=1$.

Proof. Let $\psi(x) := (1-x^2)\, T_m'(x)$, where $m := n-2$. Then

$$q(x) = \sum_{\nu=0}^{m} \frac{q(\lambda_\nu)}{\psi'(\lambda_\nu)} \frac{\psi(x)}{x-\lambda_\nu}$$

and so

$$p(x) = \sum_{\nu=0}^{m} \frac{q(\lambda_\nu)}{\psi'(\lambda_\nu)} \frac{(1-x^2)^2\, T_m'(x)}{x-\lambda_\nu}.$$

Using Lemma 9 we deduce that if ξ is a root of $\tau_n'(x)=0$, then

$$p''(\xi) = \frac{T_m(\xi)}{1-\xi^2} \sum_{\nu=0}^{m} \frac{q(\lambda_\nu)}{\psi'(\lambda_\nu)} \frac{\phi(\xi, \lambda_\nu)}{(\xi-\lambda_\nu)^4}. \tag{21}$$

In particular

$$\tau_n''(\xi) = \frac{T_m(\xi)}{1-\xi^2} \sum_{\nu=0}^{m} \frac{T_m(\lambda_\nu)}{-\lambda_\nu T_m'(\lambda_\nu) - m^2 T_m(\lambda_\nu)} \frac{\phi(\xi, \lambda_\nu)}{(\xi-\lambda_\nu)^4}$$

and since $T_m(\lambda_\nu)$ and $\psi'(\lambda_\nu) = -\lambda_\nu T'_m(\lambda_\nu) - m^2 T_m(\lambda_\nu)$ are of opposite sign this gives

$$\tau''_n(\xi) = -\frac{T_m(\xi)}{1-\xi^2} \sum_{\nu=0}^{m} \left|\frac{1}{\psi'(\lambda_\nu)}\right| \frac{\phi(\xi, \lambda_\nu)}{(\xi-\lambda_\nu)^4}. \tag{22}$$

Now $|q(\lambda_\nu)| \leqslant 1$ by hypothesis and $\phi(\xi, \lambda_\nu) \geqslant 0$ according to Lemma 12′, so comparing (21) and (22) we obtain

$$|p''(\xi)| \leqslant |\tau''_n(\xi)|$$

where equality holds if and only if $q(\lambda_\nu) = \gamma T_m(\lambda_\nu)$ $(\nu = 0, 1, ..., m)$, i.e., $p(x) \equiv \gamma \tau_n(x)$ for some constant γ, $|\gamma| = 1$.

3. Proof of Theorem 1

Let $p(x) := (1-x^2)\, q(x)$ be a polynomial of degree at most n such that $|q(\lambda_\nu)| \leqslant 1$ for $\nu = 0, 1, ..., n-2$. Further, let $p(x)$ be *real for real* x. If $p(x) \not\equiv \pm \tau_n(x)$ then by Lemma 13 there exists a constant $c > 1$ such that $|cp''(x)| \leqslant |\tau''_n(x)|$ at the zeros of τ'_n. Since the zeros of τ'_n are all real and distinct it follows from Lemma 1 that $|cp'''(x)| \leqslant |\tau'''_n(x)|$ at the zeros of τ''_n. Now Lemma 2 applied in conjunction with Lemma 4 gives

$$|p^{(k)}(x+iy)| \leqslant \frac{1}{c} |\tau_n^{(k)}(1+iy)|$$

$$\text{for} \quad (x, y) \in [-1, 1] \times \mathbb{R} \quad \text{and} \quad k = 3, 4, ...$$

Hence (12′) holds. In particular

$$\|p^{(k)}\| \leqslant |\tau_n^{(k)}(1)| \qquad \text{for} \quad k = 3, 4, ... \tag{23}$$

In this latter inequality, the condition that $p(x)$ is real for real x can be dropped. To see this, let $p(x) := (1-x^2)\, q(x)$ be a polynomial of degree at most n such that $|q(\lambda_\nu)| \leqslant 1$ for $\nu = 0, 1, ..., n-2$. Let $\|p^{(k)}\|$ be attained at $x_* \in [-1, 1]$ where $p^{(k)}(x_*) = \|p^{(k)}\|\, e^{i\alpha}$. Consider $p_*(x) := \mathrm{Re}\{e^{-i\alpha} p(x)\} = (1-x^2)\, q_*(x)$ which is a polynomial of degree at most n such that $|q_*(\lambda_\nu)| \leqslant |q(\lambda_\nu)| \leqslant 1$ for $\nu = 0, 1, ..., n-2$. Further, $p_*(x)$ is real for real x and so by (23)

$$\|p_*^{(k)}\| \leqslant \tau_n^{(k)}(1)| \qquad \text{for} \quad k = 3, 4, ...$$

But

$$\|p^{(k)}\| = e^{-i\alpha} p^{(k)}(x_*) = p_*^{(k)}(x_*) \leqslant \|p_*^{(k)}\|$$

and therefore (12) holds.

4. An Addendum to Theorem 1

Let

$$-1 =: y_0 < y_1 < \cdots < y_m := 1$$

and set

$$\omega(x) := (1+x)^{n_1} (1-x)^{n_2} \prod_{\mu=0}^{m} (x - y_\mu),$$

where n_1, n_2 are non-negative integers. Further, let

$$\omega_\mu(x) := \frac{\omega(x)}{x - y_\mu}, \qquad \mu = 0, 1, ..., m,$$

and denote by

$$\alpha_{\mu,1} \leqslant \alpha_{\mu,2} \leqslant \cdots \leqslant \alpha_{\mu,n-k}, \qquad \mu = 0, 1, ..., m$$

the zeros of $\omega_\mu^{(k)}$. Now suppose that P_n is a polynomial of degree $n := m + n_1 + n_2$ having the following properties:

(i) it has zeros of multiplicities n_1 and n_2 at -1 and 1, respectively,

(ii) the polynomial $\hat{P}_n(x) := P_n(x)/(1+x)^{n_1} (1-x)^{n_2}$ has alternating signs at the points $y_0, y_1, ..., y_m$.

It was proved in [4, Theorem 1] that if $p(x) := (1+x)^{n_1} (1-x)^{n_2} \hat{p}(x)$ is a polynomial of degree at most n such that

$$|\hat{p}(y_\mu)| \leqslant |\hat{P}_n(y_\mu)|, \qquad \mu = 0, 1, ..., m, \tag{24}$$

and $p(x)$ is real for real x then for z lying outside the open disk with $(\alpha_{m,1}, \alpha_{0,n-k})$ as diameter, we have

$$|p^{(k)}(z)| \leqslant |P_n^{(k)}(z)|.$$

The statement of Theorem 1 in [4] contains a slight inaccuracy, namely, the hats over p and P_n in (24) were inadvertently omitted.

Applying the above result with $P_n(x) := (1-x^2)\, T_{n-2}(x)$ and

$$y_\mu := -\cos\frac{\mu\pi}{n-2}, \qquad \mu = 0, 1, ..., n-2$$

we obtain

Theorem 2. *Let $p(x) := (1-x^2)\, q(x)$ be a polynomial of degree at most n such that* (11) *holds. If $p(x)$ is real for real x then for $k = 0, 1, 2, ...$*

$$|p^{(k)}(z)| \leqslant |\tau_n^{(k)}(z)| \tag{25}$$

for $|z| \geqslant \alpha_\kappa$, *where* α_κ *is the largest zero of*

$$\frac{d^k}{dx^k}\left\{\frac{(1-x^2)\,T'_{n-2}(x)}{(1+x)}\right\}.$$

According to a result in [3], inequality (25) does not hold at points *immediately* to the right of $-\alpha_\kappa$ and at those immediately to the left of α_κ. So in Theorem 2 α_κ cannot be replaced by any smaller number.

5. Some Remarks on Theorem 1

5.1. In Theorem 1 we have proved, in particular, that for $k = 3, 4, \ldots$ the conclusion of (3.1) remains true under the weaker hypothesis that $p(x)/(1-x^2)$ is bounded by 1 only at the points $x_\nu = \cos(\nu\pi/(n-2))$; $\nu = 0, 1, \ldots, n-2$. This raises the question if there are $n-1$ other points in the interval $[-1, 1]$ with the same property. The answer is in the negative. Indeed if E is any closed set of points in $[-1, 1]$ which does not include all the points $x_\nu = \cos(\nu\pi/(n-2))$ then there exists (see [1, p. 526]; also see [8, Remark 3 on p. 138]) a polynomial q of degree $n-2$ which is bounded by 1 in E whereas $q^{(k)}(1) > T^{(k)}_{n-2}(1)$ for $k = 1, 2, \ldots, n-2$. So $p(x) := (1-x^2)\,q(x)$ serves as a counter example.

$$\begin{aligned}|p^{(k)}(1)| &= 2k\,q^{(k-1)}(1) + k(k-1)\,q^{(k-2)}(1)\\ &> 2k\,T^{(k-1)}_{n-2}(1) + k(k-1)\,T^{(k-2)}_{n-2}(1) = |\tau^{(k)}_n(1)|.\end{aligned}$$

5.2. It is natural to wonder if (12′) or at least (12) holds also for $k = 2$. Further, one may ask if (3.2), (3.3) and (3.4) hold if only (11) is satisfied. The example $p(x) := (1-x^2)\,q(x)$, where $q(x) := -x^2 + x + 1$, shows that (3.2) does not hold under the weaker assumption. Indeed $|q(\cos(\nu\pi/2))| = 1$ for $\nu = 0, 1, 2$ whereas $\|p'\| = (9 + 19\sqrt{57})/72 > 2 = |\tau'_4(1)|$. The other parts of the question will be discussed elsewhere.

5.3. Theorem 1 may also be stated as follows.

Theorem 1′. *If* p *is a polynomial of degree at most* n *satisfying* (4) *then*

$$\left\|\frac{d^k}{dx^k}((1-x^2)\,p(x))\right\| \leqslant |\tau^{(k)}_{n+2}(1)| \qquad \textit{for} \quad k = 3, 4, \ldots.$$

Further, if $p(x)$ *is real for real* x, *and* $I_y := \{x + iy\colon -1 \leqslant x \leqslant 1\}$ *then*

$$\max_{z \in I_y}\left|\frac{d^k}{dz^k}((1-z^2)\,p(z))\right| \leqslant |\tau^{(k)}_{n+2}(1+iy)| \qquad \textit{for} \quad y \in \mathbb{R} \quad \textit{and } k = 3, 4, \ldots.$$

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